TITLE:

Attractive Point and Weak Convergence Theorems for Two Commutative Nonlinear Mappings in Banach Spaces (The deepening of function spaces and its environment)

AUTHOR(S):

Takahashi, Wataru

CITATION:

Takahashi, Wataru. Attractive Point and Weak Convergence Theorems for Two Commutative Nonlinear Mappings in Banach Spaces (The deepening of function spaces and its environment). 数理解析研究所講究録 2018, 2095: 114-123

ISSUE DATE:

2018-12

URL:

http://hdl.handle.net/2433/251712

RIGHT:

京都大学学術情報リポジトリ KURENAI 紅 Kyoto University Research Information Repository

数理解析研究所講究録 第2095巻 2018年 114-123

Attractive Point and Weak Convergence Theorems for Two Commutative Nonlinear Mappings in Banach Spaces

慶応義塾大学自然科学研究教育センター, 高雄医学大学基礎科学センター
高橋渉 (Wataru Takahashi)
Keio Research and Education Center for Natural Sciences, Keio University, Japan and
Center for Fundamental Science, Kaohsiung Medical University, Kaohsiung 80702, Taiwan
Email: wataru@is.titech.ac.jp; wataru@a00.itscom.net

Abstract. In this article, using the class of generalized nonspreading mappings in Banach spaces which covers generalized hybrid mappings in a Hilbert space, we prove an attractive point theorem. Furthermore, we prove a nonlinear mean convergence theorem of Baillon's type and a weak convergence theorem of Mann's type for generalized nonspreading mappings in a Banach space. Using these theorems, we obtain new attractive point theorems, mean convergence theorems and weak convergence theorems in Hilbert spaces and Banach spaces.

2010 *Mathematics Subject Classification*: 47H05, 47H09
Keywords and phrases: Banach space, generalized hybrid mapping, generalized nonspreading mapping, fixed point, attractive point, nonlinear ergodic theorem.

1 Introduction

Let H be a real Hilbert space and let C be a nonempty subset of H. Let T be a mapping of C into H. Then we denote by $F(T)$ the set of *fixed points* of T and by $A(T)$ the set of *attractive points* [33] of T, i.e.,

(i) $F(T) = \{z \in C : Tz = z\}$;
(ii) $A(T) = \{z \in H : \|Tx - z\| \leq \|x - z\|, \quad \forall x \in C\}$.

We know from [33] that $A(T)$ is closed and convex. This property is important for proving our main theorems. In 2010, Kocourek, Takahashi and Yao [17] defined a broad class of nonlinear mappings in a Hilbert space: Let H be a real Hilbert space and let C be a nonempty subset of H. A mapping $T : C \to H$ is called generalized hybrid [17] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \tag{1.1}$$

for all $x, y \in C$. Such a mapping T is called (α, β)-generalized hybrid. We also know the following mapping: For $\lambda \in \mathbb{R}$, $U : C \to H$ is called λ-hybrid [2] if

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 + 2(1-\lambda)\langle x - Ux, y - Uy\rangle \tag{1.2}$$

for all $x, y \in C$. Notice that the class of generalized hybrid mappings covers several well-known mappings. For example, a (1,0)-generalized hybrid mapping is nonexpansive. It is nonspreading [21, 22] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

It is also hybrid [31] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

In general, nonspreading and hybrid mappings are not continuous; see [14]. We know that λ-hybrid mappings are cotained in the class of generalized hybrid mappings; see [7]. In 1975, Baillon [3] proved the following nonlinear ergodic theorem in a Hilbert space:

Theorem 1.1 ([3]). *Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. Let $T : C \to C$ be a nonexpansive mapping such that the set $F(T)$ of fixed points of T is nonempty. Then, for any $x \in C$,*

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to a point of $F(T)$.

This theorem for nonexpansive mappings has been extended to Banach spaces by many authors; see, for example, [4, 5, 6, 23, 24]. On the other hand, Kocourek, Takahashi and Yao [17] extended this theorem to generalized hybrid mappings in a Hilbert space. Recently, Kohsaka [19] also proved the following theorem:

Theorem 1.2 ([19]). *Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. Let S and T be commutative λ and μ-hybrid mappings of C into itself such that the set $F(S) \cap F(T)$ of common fixed points of S and T is nonempty. Then, for any $x \in C$,*

$$S_n x = \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x$$

converges weakly to a point of $F(S) \cap F(T)$.

We also know Mann's iteration [26] introduced in 1953. Let C be a nonempty, closed and convex subset of a Banach space E. A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. For an initial guess $x_1 \in C$, an iteration process $\{x_n\}$ is defined recursively by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$.

In this article, using the class of generalized nonspreading mappings in Banach spaces which covers generalized hybrid mappings in a Hilbert space, we prove an attractive point theorem. Furthermore, we prove a nonlinear mean convergence theorem of Baillon's type and a weak convergence theorem of Mann's type for generalized nonspreading mappings in a Banach space. Using these theorems, we obtain new attractive point theorems, mean convergence theorems and weak convergence theorems in Hilbert spaces and Banach spaces.

2 Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the topological dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \epsilon \right\}$$

for every ϵ with $0 \le \epsilon \le 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. A uniformly convex Banach space is strictly convex and reflexive. Let C be a nonempty subset of a Banach space E. A mapping $T : C \to E$ is nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to E$ is quasi-nonexpansive if $F(T) \ne \emptyset$ and $\|Tx - y\| \le \|x - y\|$ for all $x \in C$ and $y \in F(T)$, where $F(T)$ is the set of fixed points of T. If C is a nonempty, closed and convex subset of a strictly convex Banach space E and $T : C \to E$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see Itoh and Takahashi [15]. Let E be a Banach space. The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists. In this case, E is called smooth. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in U$, the limit (2.1) is attained uniformly for $x \in U$. It is also said to be Fréchet differentiable if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$. A Banach space E is called uniformly smooth if the limit (2.1) is attained uniformly for $x, y \in U$. It is known that if the norm of E is uniformly Gâteaux differentiable, then J is uniformly norm-to-weak* continuous on each bounded subset of E, and if the norm of E is Fréchet differentiable, then J is norm-to-norm continuous. If E is uniformly smooth, J is uniformly norm-to-norm continuous on each bounded subset of E. For more details, see [29, 30]. The following result is also well known.

Lemma 2.1 ([29]). *Let E be a smooth Banach space and let J be the duality mapping on E. Then, $\langle x-y, Jx-Jy \rangle \ge 0$ for all $x, y \in E$. Further, if E is strictly convex and $\langle x-y, Jx-Jy \rangle = 0$, then $x = y$.*

Let E be a smooth Banach space. The function $\phi \colon E \times E \to (-\infty, \infty)$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

for $x, y \in E$, where J is the duality mapping of E; see [1] and [16]. We have from the definition of ϕ that

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle \tag{2.2}$$

for all $x, y, z \in E$. From $(\|x\| - \|y\|)^2 \leq \phi(x, y)$ for all $x, y \in E$, we can see that $\phi(x, y) \geq 0$. Furthermore, we can obtain the following equality:

$$2\langle x - y, Jz - Jw\rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w) \tag{2.3}$$

for $x, y, z, w \in E$. If E is additionally assumed to be strictly convex, then

$$\phi(x, y) = 0 \Longleftrightarrow x = y. \tag{2.4}$$

The following lemmas are in Xu [36] and Kamimura and Takahashi [16].

Lemma 2.2 ([36])**.** *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|)$$

for all $x, y \in B_r$ and λ with $0 \leq \lambda \leq 1$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Lemma 2.3 ([16])**.** *Let E be smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and*

$$g(\|x - y\|) \leq \phi(x, y)$$

for all $x, y \in B_r$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Let E be a smooth Banach space and let C be a nonempty subset of E. Then a mapping $T : C \to E$ is called generalized nonexpansive [10] if $F(T) \neq \emptyset$ and

$$\phi(Tx, y) \leq \phi(x, y)$$

for all $x \in C$ and $y \in F(T)$. Let D be a nonempty subset of a Banach space E. A mapping $R : E \to D$ is said to be sunny if

$$R(Rx + t(x - Rx)) = Rx$$

for all $x \in E$ and $t \geq 0$. A mapping $R : E \to D$ is said to be a retraction or a projection if $Rx = x$ for all $x \in D$. A nonempty subset D of a smooth Banach space E is said to be a generalized nonexpansive retract (resp. sunny generalized nonexpansive retract) of E if there exists a generalized nonexpansive retraction (resp. sunny generalized nonexpansive retraction) R from E onto D; see [9, 10] for more details. The following results are in Ibaraki and Takahashi [10].

Lemma 2.4 ([10])**.** *Let C be a nonempty closed sunny generalized nonexpansive retract of a smooth and strictly convex Banach space E. Then the sunny generalized nonexpansive retraction from E onto C is uniquely determined.*

Lemma 2.5 ([10])**.** *Let C be a nonempty closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C and let $(x, z) \in E \times C$. Then the following hold:*

(i) $z = Rx$ if and only if $\langle x - z, Jy - Jz\rangle \leq 0$ for all $y \in C$;
(ii) $\phi(Rx, z) + \phi(x, Rx) \leq \phi(x, z)$.

In 2007, Kohsaka and Takahashi [20] proved the following results:

Lemma 2.6 ([20]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed subset of E. Then the following are equivalent:*

(a) C is a sunny generalized nonexpansive retract of E;
(b) C is a generalized nonexpansive retract of E;
(c) JC is closed and convex.

Lemma 2.7 ([20]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed sunny generalized nonexpansive retract of E. Let R be the sunny generalized nonexpansive retraction from E onto C and let $(x, z) \in E \times C$. Then the following are equivalent:*

(i) $z = Rx$;
(ii) $\phi(x, z) = \min_{y \in C} \phi(x, y)$.

Ibaraki and Takahashi [13] also obtained the following result concerning the set of fixed points of a generalized nonexpansive mapping.

Lemma 2.8 ([13]). *Let E be a smooth, strictly convex and reflexive Banach space and let T be a generalized nonexpansive mapping from E into itself. Then, $F(T)$ is closed and $JF(T)$ is closed and convex.*

The following lemma is a direct consequence of Lemmas 2.6 and 2.8.

Lemma 2.9 ([13]). *Let E be a smooth, strictly convex and reflexive Banach space and let T be a generalized nonexpansive mapping from E into itself. Then, $F(T)$ is a sunny generalized nonexpansive retract of E.*

Using Lemma 2.6, we have the following result.

Lemma 2.10. *Let E be a smooth, strictly convex and reflexive Banach space and let $\{C_i : i \in I\}$ be a family of sunny generalized nonexpansive retracts of E such that $\cap_{i \in I} C_i$ is nonempty. Then $\cap_{i \in I} C_i$ is a sunny generalized nonexpansive retract of E.*

3 Attractive Point and Fixed Point Theorem

Kocourek, Takahashi and Yao [18] extended the concept of generalized hybrid mappings [17] in a Hilbert space to that in a Banach space. Let E be a smooth Banach space and let C be a nonempty subset of E. Then a mapping $T : C \to E$ is called *generalized nonspreading* [18] if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\alpha\phi(Tx, Ty) + (1 - \alpha)\phi(x, Ty) + \gamma\{\phi(Ty, Tx) - \phi(Ty, x)\} \\ \leq \beta\phi(Tx, y) + (1 - \beta)\phi(x, y) + \delta\{\phi(y, Tx) - \phi(y, x)\}$$

for all $x, y \in C$. We call such a mapping $(\alpha, \beta, \gamma, \delta)$-generalized nonspreading. Let E be a smooth Banach space. Let C be a nonempty subset of E and let T be a mapping of C into E. We denote by $A(T)$ the set of *attractive points* of T, i.e., $A(T) = \{z \in E : \phi(z, Tx) \leq \phi(z, x), \ \ \forall x \in C\}$; see [25].

Lemma 3.1 ([25]). *Let E be a smooth Banach space and let C be a nonempty subset of E. Let T be a mapping from C into E. Then $A(T)$ is a closed and convex subset of E.*

We prove the following lemma.

Lemma 3.2. *Let E be a smooth, strictly convex and reflexive Banach space with the duality mapping J and let C be a nonempty subset of E. Let S and T be mappings of C into itself. Let $\{x_n\}$ be a bounded sequence of E and let μ be a mean on l^∞. Suppose that*

$$\mu_n \phi(x_n, Sy) \le \mu_n \phi(x_n, y) \text{ and } \mu_n \phi(x_n, Ty) \le \mu_n \phi(x_n, y)$$

for all $y \in C$. Then $A(S) \cap A(T)$ is nonemppty. Additionally, if C is closed and convex and $\{x_n\} \subset C$, then $F(S) \cap F(T)$ is nonempty.

Using Lemma 3.2, we can prove an attractive point and fixed point theorem for commutative generalized nonspreading mappings in a Banach space.

Theorem 3.3 ([34])**.** *Let C be a nonempty subset of a smooth, strictly convex and reflexive Banach space E and let S and T be commutative generalized nonspreading mappings of C into itself. Suppose that there exists an element $z \in C$ such that $\{S^k T^l z : k, l \in \mathbb{N} \cup \{0\}\}$ is bounded. Then $A(S) \cap A(T)$ is nonempty. Additionally, if C is closed and convex, then $F(S) \cap F(T)$ is nonempty.*

4 Nonlinear Ergodic Theorems of Baillon's Type

Now, using the technique developed by [28], we prove a mean convergence theorem of Baillon's type for generalized nonspreading mappings in a Banach space. For proving it, we need the following lemma.

Lemma 4.1. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty, closed and convex subset of E. Let S and T be commutative generalized nonspreading mappings of C into itself. If $\{S^k T^l x : k, l \in \mathbb{N} \cup \{0\}\}$ for some $x \in C$ is bounded and*

$$S_n x = \frac{1}{(1+n)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x$$

for all $n \in \mathbb{N} \cup \{0\}$, then every weak cluster point of $\{S_n x\}$ is a point of $F(S) \cap F(T)$.

Let E be a smooth Banach space. Let C be a nonempty subset of E and let T be a mapping of C into E. We denote by $B(T)$ the set of *skew-attractive points* [25] of T, i.e., $B(T) = \{z \in E : \phi(Tx, z) \le \phi(x, z), \quad \forall x \in C\}$. Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty subset of E. Let T be a mapping of C into E. Define a mapping T^* as follows:

$$T^* x^* = JTJ^{-1} x^*, \quad \forall x^* \in JC,$$

where J is the duality mapping on E and J^{-1} is the duality mapping on E^*. A mapping T^* is called the duality mapping of T; see also [35] and [8]. It is easy to show that if T is a mapping of C into itselt, then T^* is a mapping of JC into itself. In fact, for $x^* \in JC$, we have $J^{-1}x^* \in C$ and hence $TJ^{-1}x^* \in C$. So, we have

$$T^* x^* = JTJ^{-1} x^* \in JC.$$

Then, T^* is a mapping of JC into itself. The following result is in Lin and Takahashi [25].

Lemma 4.2 ([25]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty subset of E. Let T be a mapping of C into E and let T^* be the duality mapping of T. Then, the following hold:*

(1) $JB(T) = A(T^*)$;
(2) $JA(T) = B(T^*)$.

In particular, $JB(T)$ is closed and convex.

Let $D = \{(k,l) : k,l \in \mathbb{N} \cup \{0\}\}$. Then D is a directed set by the binary relation:

$$(k,l) \leq (i,j) \quad \text{if} \quad k \leq i \ \text{ and } \ l \leq j.$$

Now, we can prove the following nonlinear ergodic theorem for generalized nonspreading mappings in a Banach space.

Theorem 4.3 ([32]). *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty, closed and convex sunny generalized nonexpansive retract of E. Let S and T be commutative generalized nonspreading mappings of C into itself with $F(S) \cap F(T) \neq \emptyset$ such that $\phi(Sx,u) \leq \phi(x,u)$ and $\phi(Tx,v) \leq \phi(x,v)$ for all $x \in C$ and $u \in F(S)$ and $v \in F(T)$, respectively. Let R be the sunny generalized nonexpansive retraction of E onto $F(S) \cap F(T)$. Then, for any $x \in C$,*

$$S_n x = \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x$$

converges weakly to an element q of $F(S) \cap F(T)$, where $q = \lim_{(k,l)\in D} RS^kT^l x$.

Using Theorem 4.3, we obtain the two following theorems.

Theorem 4.4. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm. Let $S,T : E \to E$ be commutative $(\alpha,\beta,\gamma,\delta)$ and $(\alpha',\beta',\gamma',\delta')$-generalized nonspreading mappings with $F(S) \cap F(T)$ such that $\alpha > \beta$ and $\gamma \leq \delta$ and $\alpha' > \beta'$ and $\gamma' \leq \delta'$, respectively. Assume that $F(S) \cap F(T) \neq \emptyset$ and let R be the sunny generalized nonexpansive retraction of E onto $F(S) \cap F(T)$. Then, for any $x \in E$,*

$$S_n x = \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x$$

converges weakly to an element q of $F(S) \cap F(T)$, where $q = \lim_{(k,l)\in D} RS^kT^l x$.

Theorem 4.5. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $S,T : C \to C$ be commutative generalized hybrid mappings with $F(S) \cap F(T) \neq \emptyset$ and let P be the mertic projection of H onto $F(S) \cap F(T)$. Then, for any $x \in C$,*

$$S_n x = \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x$$

converges weakly to an element p of $F(S) \cap F(T)$, where $p = \lim_{(k,l)\in D} PS^kT^l x$.

5 Weak Convergence Theorems of Mann's Type

In this section, we prove a weak convergence theorem of Mann's type iteration for generalized nonspreading mappings in a Banach space. For proving it, we need the following lemma.

Lemma 5.1. *Let E be a smooth and uniformly convex Banach space and let C be a nonempty and closed subset of E such that JC is closed and convex. Let S and T be commutative generalized nonspreading mappings of C into itself such that $F(S) \cap F(T) \neq \emptyset$, $\phi(Sx, u) \leq \phi(x, u)$ and $\phi(Tx, v) \leq \phi(x, v)$ for for all $x \in C$ and $u \in F(S)$ and $v \in F(T)$. Let R be a sunny generalized nonexpansive retraction of E onto $F(S) \cap F(T)$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and let $\{x_n\}$ be a sequence in C generated by $x_1 = x \in C$ and*

$$x_{n+1} = R_C\Big(\alpha_n x_n + (1 - \alpha_n)\frac{1}{(n+1)^2}\sum_{k=0}^{n}\sum_{l=0}^{n} S^k T^l x_n\Big), \quad \forall n \in \mathbb{N},$$

where R_C is a sunny generalized nonexpansive retraction of E onto C. Then $\{Rx_n\}$ converges strongly to a point z of $F(S) \cap F(T)$.

Using Lemma 5.1 and the technique developed by [11], we prove the following theorem.

Theorem 5.2 ([32]). *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty closed convex sunny generalized nonexpansive retract of E. Let S and T be commutative generalized nonspreading mappings of C into itself such that $F(S) \cap F(T) \neq \emptyset$, $\phi(Sx, u) \leq \phi(x, u)$ and $\phi(Tx, v) \leq \phi(x, v)$ for for all $x \in C$ and $u \in F(S)$ and $v \in F(T)$. Let R be the sunny generalized nonexpansive retraction of E onto $F(S) \cap F(T)$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. Then, a sequence $\{x_n\}$ generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)\frac{1}{(n+1)^2}\sum_{k=0}^{n}\sum_{l=0}^{n} S^k T^l x_n, \quad \forall n \in \mathbb{N}$$

converges weakly to $z \in F(S) \cap F(T)$, where $z = \lim_{n\to\infty} Rx_n$.

Using Theorem 5.2, we can prove the following two weak convergence theorems.

Theorem 5.3. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm. Let $S, T : E \to E$ be commutative $(\alpha, \beta, \gamma, \delta)$ and $(\alpha', \beta', \gamma', \delta')$-generalized nonspreading mappings such that $\alpha > \beta$ and $\gamma \leq \delta$ and $\alpha' > \beta'$ and $\gamma' \leq \delta'$, respectively. Assume that $F(S) \cap F(T) \neq \emptyset$ and let R be the sunny generalized nonexpansive retraction of E onto $F(S) \cap F(T)$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. Then, a sequence $\{x_n\}$ generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)\frac{1}{(n+1)^2}\sum_{k=0}^{n}\sum_{l=0}^{n} S^k T^l x_n x_n, \quad \forall n \in \mathbb{N}$$

converges weakly to $z \in F(S) \cap F(T)$, where $z = \lim_{n\to\infty} Rx_n$.

Theorem 5.4. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $S, T : C \to C$ be commutative generalized hybrid mappings with $F(S) \cap F(T) \neq \emptyset$ and let P be the mertic projection of H onto $F(S) \cap F(T)$. Let $\{\alpha_n\}$ be a sequence of real numbers* *such that* $0 \leq \alpha_n < 1$ *and* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. *Then, a sequence* $\{x_n\}$ *generated by* $x_1 = x \in C$ *and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\frac{1}{(n+1)^2}\sum_{k=0}^{n}\sum_{l=0}^{n} S^k T^l x_n x_n, \quad \forall n \in \mathbb{N}$$

converges weakly to $z \in F(S) \cap F(T)$, *where* $z = \lim_{n\to\infty} P x_n$.

Acknowledgements. The author was partially supported by Grant-in-Aid for Scientific Research No. 15K04906 from Japan Society for the Promotion of Science.

References

[1] Y. I. Alber, *Metric and generalized projections in Banach spaces: Properties and applications*, in Theory and Applications of Nonlinear Operators of Accretive and Monotone Type (A. G. Kartsatos Ed.), Marcel Dekker, New York, 1996, pp. 15–50.

[2] K. Aoyama, S. Iemoto, F. Kohsaka and W. Takahashi, *Fixed point and ergodic theorems for λ-hybrid mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **11** (2010), 335-343.

[3] J.-B. Baillon, *Un theoreme de type ergodique pour les contractions non lineaires dans un espace de Hilbert,* C.R. Acad. Sci. Paris Ser. A-B **280** (1975), 1511-1514.

[4] R. E. Bruck, Jr, *A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces*, Israel J. Math. **32** (1979), 107–116.

[5] N. Hirano, K. Kido and W. Takahashi, *The existence of nonexpansive retractions in Banach spaces*, J. Math. Soc. Japan **38** (1986), 1–7.

[6] N. Hirano and W. Takahashi, *Nonlinear ergodic theorems for an amenable semigroup of nonexpansive mappings in a Banach space*, Pacific J. Math. **112** (1984), 333–346.

[7] M. Hojo, W. Takahashi and J.-C.Yao, *Weak and strong convergence theorems for supper hybrid mappings in Hilbert spaces*, Fixed Point Theory **12** (2011), 113–126.

[8] T. Honda, T. Ibaraki and W. Takahashi, *Duality theorems and convergence theorems for nonlineaqr mappings in Banach spaces*, Int. J. Math. Statis. **6** (2010), 46–64.

[9] T. Ibaraki and W. Takahashi, *Mosco convergence of sequences of retracts of four nonlinear projections in Banach spaces*, in Nonlinear Analysis and Convex Analtsis (W. Takahashi and T. Tanaka Eds.), Yokohama Publishers, Yokohama, 2007, pp. 139–147.

[10] T. Ibaraki and W. Takahashi, *A new projection and convergence theorems for the projections in Banach spaces*, J. Approx. Theory **149** (2007), 1–14.

[11] T. Ibaraki and W. Takahashi, *Weak convergence theorem for new nonexpansive mappings in Banach spaces and its applications*, Taiwanese J. Math. **11** (2007), 929–944.

[12] T. Ibaraki and W. Takahashi, *Fixed point theorems for new nonlinear mappings of nonexpansive type in Banach spaces*, J. Nonlinear Convex Anal. **10** (2009), 21–32.

[13] T. Ibaraki and W. Takahashi, *Generalized nonexpansive mappings and a proximal-type algorithm in Banach spaces*, Nonlinear analysis and optimization I. Nonlinear analysis, 169–180, Contemp. Math., 513, Amer. Math. Soc., Providence, RI, 2010.

[14] T. Igarashi, W. Takahashi and K. Tanaka, *Weak convergence theorems for nonspreading mappings and equilibrium problems*, in Nonlinear Analysis and Optimization (S. Akashi, W. Takahashi and T. Tanaka Eds.), Yokohama Publishers, Yokohama, 2008, pp. 75–85.

[15] S. Itoh and W. Takahashi, *The common fixed point theory of singlevalued mappings and multivalued mappings*, Pacific J. Math. **79** (1978), 493–508.

[16] S. Kamimura and W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach apace*, SIAM J. Optim. **13** (2002), 938–945.

[17] P. Kocourek, W. Takahashi and J. -C. Yao, *Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert spaces*, Taiwanese J. Math. **14** (2010), 2497–2511.

[18] P. Kocourek, W. Takahashi and J. -C. Yao, *Fixed point theorems and ergodic theorems for nonlinear mappings in Banach spaces*, Adv. Math. Econ. **15** (2011), 67–88.

[19] F. Kohsaka, *Existence and approximation of common fixed points of two hybrid mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **16** (2015), 2193-2205.

[20] F. Kohsaka and W. Takahashi, *Generalized nonexpansive retractions and a proximal-type algorithm in Banach spaces*, J. Nonlinear Convex Anal. **8** (2007), 197-209.

[21] F. Kohsaka and W. Takahashi, *Existence and approximation of fixed points of firmly nonexpansive-type mappings in Banach spaces*, SIAM J. Optim. **19** (2008), 824-835.

[22] F. Kohsaka and W. Takahashi, *Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces*, Arch. Math. **91** (2008), 166-177.

[23] A. T. Lau, N. Shioji and W. Takahashi, *Existence of nonexpansive retractions for amenable semigroups of nonexpansive mappings and nonlinear ergodic theorems in Banach spaces*, J. Funct. Anal. **161** (1999), 62–75.

[24] A. T. Lau and W. Takahashi, *Weak convergence and nonlinear ergodic theorems for reversible semigroups of nonexpansive mappings*, Pacific J. Math. **126** (1987), 277–294.

[25] L.-J. Lin and W. Takahashi, *Attractive point theorems for generalized nonspreading mappings in Banach spaces*, J. Convex Anal. **20** (2013), 265–284.

[26] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506-510.

[27] S. Matsushita and W. Takahashi, *A strong convergence theorem for relatively nonexpansive mappings in a Banach space*, J. Approx. Theory **134** (2005), 257–266.

[28] W. Takahashi, *A nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hilbert space*, Proc. Amer. Math. Soc. **81** (1981), 253–256.

[29] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, 2000.

[30] W. Takahashi, *Convex Analysis and Approximation of Fixed Points (Japanese)*, Yokohama Publishers, Yokohama, 2000.

[31] W. Takahashi, *Fixed point theorems for new nonexpansive mappings in a Hilbert space*, J. Nonlinear Convex Anal. **11** (2010), 79–88.

[32] W. Takahashi, *Weak convergence theorems for two generalized nonspreading mappings in Banach spaces*, J. Nonlinear Convex Anal. **18** (2017), 1207–1223.

[33] W. Takahashi and Y. Takeuchi, *Nonlinear ergodic theorem without convexity for generalized hybrid mappings in a Hilbert space*, J. Nonlinear Convex Anal. **12** (2011), 399–406.

[34] W. Takahashi, Ching-Feng Wen and Jen-Chih Yao, *Attractive and mean convergence theorems for two commutative nonlinear mappings in Banach spaces*, Dynamical Systems and Applications **26** (2017), 327–346.

[35] W. Takahashi and J.-C. Yao, *Nonlinear operators of monotone type and convergence theorems with equilibrium problems in Banach spaces*, Taiwanese J. Math. **15** (2011), 787–818.

[36] H. K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal. **16** (1981), 1127–1138.